NORTH-HOLLAND

On the Schmidt Pairs of Multivariable Hankel Operators and Robust Control

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Dedicated to M. Fiedler and V. Pták.

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ABSTRACT

We study here a particular application of the theory of Hankel operators to robust control. The techniques of superoptimal Nehari extension developed by Young are employed to derive some properties of a particular controller of a rational function G, called the superoptimally robust stabilizing controller. Using this controller, we generalize to the multivariable case some results due to Fuhrmann and Ober. We also give a characterization of all plants G which are stabilized by the same superoptimally robust stabilizing controller.

1. INTRODUCTION

The relevance of Hankel operators in the theory of robust control has been apparent since the seminal paper of Glover [7], and although its foundations go back to the work of Adamjan, Arov, and Krein [1], some of its features have been studied in detail only recently: in [6] for relations between Schmidt pairs and robust control, and by Young (see [16]) for a generalization of the basic idea of [1] to the multivariable case. The work of Young, in

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particular, has not been exploited very much (in [14] and [9] this problem, is studied from a realization approach). Still, we believe that there is some insight to be gained by using Young's approach to extend some results of [6] to the multivariable case. In particular, the characterization of the superoptimal controller (at the beginning of Section 3) is slightly simpler than the usual one. Moreover, with this characterization it becomes quite natural to pose an inverse problem: what are the rational functions which have the same superoptimally robust stabilizing controller? These are, in fact, the basic contributions of this work.

The problem we consider is the following: We are given a rational function G of dimension $p \times m$ and an internally stabilizing controller K of G (that is, a rational function K such that

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}$$

is a unit in the Hardy space H_+^∞); can we relate the (conjugate) Hankel operator with symbol $[M, N]$ to the Hankel operator with symbol $M^*U + N^*V$? What can be said about singular values and Schmidt pairs? It is known that there is a "best" controller (U_Λ, V_Λ) in some sense, called the superoptimally robust stabilizing controller. Can we characterize all functions which have the same "best" controller? The reason why these questions have some interest is that there are some intrinsic invariants [the inner functions Q and $\overline{Q}$ defined in (4)], which are more easily expressed in terms of the symbol $M^*U + N^*V$ rather than $[M, N]$, and also the inverse problem is more easily expressed in terms of these functions.

We show, in particular, that the Schmidt pairs of the (conjugate) Hankel operator with symbol $[M, N]$ can be expressed in terms of the Schmidt vectors of the Hankel operator with symbol $M^*U + N^*V$ [this operator is independent of the choice of (U, V) in the class of stabilizing controllers], and of the superoptimally robust stabilizing controller (U_Λ, V_Λ). The name comes from the fact that to determine this controller it is sufficient to derive a particular Nehari extension R_Λ of the antistable part R_0^* of $M^*U + N^*V$, which has the property that $\sup_{\omega \in \mathbf{R}} \sigma_i(R_\Lambda)(i\omega)$ is minimal for each i, $1 \leqslant i \leqslant p$.

The reason why so much attention has been devoted, in the control community, to the study of the (conjugate) Hankel operator with symbol $[M, N]$ is that it can be used to deal with a certain degree of *uncertainty* in the choice of the rational function to stabilize: under certain conditions, the controller K which stabilizes G also stabilizes a class of rational functions which depends on the pair (M, N); an important contribution was given by Vidyasagar and Kimura [15], and their result is quoted below.

We believe that a detailed analysis of the Schmidt pairs of the above operators can be relevant for approximation. In fact, most approximation techniques for rational functions in control theory make use of the singular values and singular vectors.

The paper is structured as follows: Section 2 begins with some notation, and then gives a brief account of the results of [16] adjusted to our setting; Section 3 contains the main results about the relation between the two operators mentioned above (Theorem 4). In Section 4 we examine the rational functions which have the same optimally robust stabilizing controller.

2. PRELIMINARIES AND NOTATION

We work in the Hilbert space setting of the plane; we define [8] $L^2(\mathbb{I})$ to be the set of the vector or matrix valued (the proper dimension will be clear from the context) square integrable functions on the imaginary axis, and H^2_+ to be the subspace of L^2 of functions analytic in the right half-plane and such that

$$\sup_{x>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}[F^*(x+iy)F(x+iy)]\,dy < \infty$$

where * denotes transposed conjugate. If F and G are vectors, the inner product in H^2_+ is

$$\langle F, G\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(i\omega)F(i\omega)\,d\omega.$$

Analogously, H^∞_+ is the subspace of L^2 of functions analytic in the right half plane and such that

$$\sup_{\operatorname{Re} s>0} \|F(s)\| < \infty$$

where $\|F(x+iy)\|$ denotes the usual matrix norm. (H^2_- and H^∞_- are defined similarly on the left half plane.)

Let F be function of L^∞: we denote by $\mathbf{P}_+$ ($\mathbf{P}_-$) the orthogonal projection of L^2 onto H^2_+ (H^2_-). The Hankel operator with symbol F is defined as

$$H_F h = \mathbf{P}_- F h, \qquad h \in H_+^2. \tag{1}$$

Observe that, for $s \leqslant 0$, we have $H_F e^{i\omega s} h = \mathbf{P}_- e^{i\omega s} H_F h$. In a symmetric manner we can define the *conjugate* Hankel operator $\hat{H}_F$ by

$$\hat{H}_F h = \mathbf{P}_+ F k, \qquad k \in H_+^2.$$

A Schmidt pair (ξ, η) of H_F is a pair of vectors $\xi \in H_+^2$, $\eta \in H_-^2$ such that

$$H_F \xi = \sigma \eta, \qquad H_F^* \eta = \sigma \xi \tag{2}$$

for a suitable positive number σ, called a singular value.

From now on we assume all the functions to be rational. It can then be shown (see [1] or [5]) that if the rational function F is in H_-^2 and has degree n, then there exist at most n positive singular values

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$$

and n linearly independent pairs $(\xi_1, \eta_1), \dots, (\xi_n, \eta_n)$ (Schmidt pairs) satisfying (2).

By $s_i(A)$ we denote the ith singular value of a matrix A. If F is an $m \times p$ matrix, with $p \leqslant m$ in L^∞, we set, following [16],

$$s_i^\infty(F) := \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} s_i(F(i\omega)), \qquad 1 \leqslant i \leqslant p.$$

Let $R_0^* \in H_-^\infty$, and define

$$\lambda_1 := \inf_{S \in H_+^2} s_1^\infty(R_0^* + S)$$

and recursively

$$\lambda_{:=} \inf_{S \in \mathcal{S}_{i-1}} s_i^\infty(R_0^* + S),$$

where we have set $\mathcal{S}_i = \{S \in H_+^2;\ s_j^\infty(R_0^* + S) = \lambda_j,\ 1 \leqslant j \leqslant i\}$. The matrix $\operatorname{diag}\{\lambda_1, \dots, \lambda_p\}$ will be denoted by Λ. A function $R_\Lambda^* \in L^\infty$ is called superoptimal Nehari extension of a function $R_0^* \in H_-^\infty$ if

(a) $H_{R_0^*} = H_{R_\Lambda^*}$;
(b) the function R_Λ^* satisfies the following minimal property:

$$s_i^\infty(R_\Lambda^*) = \lambda_i, \qquad 1 \leqslant i \leqslant p.$$

The first property is equivalent, as is well known, to the fact that the strictly proper antistable parts of the functions coincide.

By $A^\#$ we denote the Moore-Penrose inverse of a matrix A. We can now quote a theorem from [11]:

THEOREM 1. *Let R_0^* be a $m \times p$ rational matrix-valued function in H_-^∞, with $p \leqslant m$. Then there exists a unique function $R_\Lambda^* \in L^\infty$ which is a superoptimal Nehari extension of R_0^*, and it can be written as*

$$R_\Lambda^* = \sum_{i=1}^{p} \lambda_i y_i x_i^\#, \tag{3}$$

where x_i and y_i, $i = 1, \dots, p$, are vectors in L^2 such that $\|x_i\|_2 = \|y_i\|_2$ and x_i is pointwise orthogonal to x_j for $i \neq j$ (and similarly y_i is pointwise orthogonal to y_j for $i \neq j$). Moreover, the vectors x and y are such that $x_i^*(i\omega)x_i(i\omega) = y_i^*(i\omega)y_i(i\omega)$ for a.a. ω.

Observe that the matrix dimension of the plant was $m \times p$; we have therefore inverted the dimension for R_Λ^*; the reason will be apparent in the sequel.

It follows easily from pointwise orthogonality and pointwise equality of the norms that we can write

$$R_\Lambda^* = \overline{Q}^* D^* \Lambda Q^* \tag{4}$$

where Q and $\overline{Q}$ are inner, D^* is diagonal all-pass, and Λ is constant diagonal. We refer to the values of the matrix Λ as *Young values* of the function.

REMARK. This result was originally formulated in [16]. Our formulation is more similar to the one in [11], even though our setting is continuous and not discrete. The original formulation requires x_i and y_i to be unit vectors, and so the pseudoinverse $x^\#$ coincides with the transposed conjugate x^*.

We now turn to a generic Schmidt pair: let σ_k and ξ_k, η_k denote, respectively, the kth singular value and Schmidt pair of $H_{R_\Lambda^*}$, for $1 \leqslant k \leqslant n$.

First of all, it is obvious that there exist unique $\psi_k \in H_+^2$ and $\phi_k \in H_-^2$ such that

$$R_\Lambda^* \xi_k = \sigma_k \eta_k + \psi_k, \tag{5}$$

$$R_\Lambda \eta_k = \sigma_k \xi_k + \phi_k. \tag{6}$$

The above are called, following [5], *fundamental equations* for $H_{R_\Lambda^*}$. They clearly characterize the Schmidt pairs.

Let a rational matrix $G \in L^\infty$ of dimension $p \times m$ be given. We say that a factorization NM^{-1} of G is a normalized right coprime factorization (NRCF) if M, N are in H_+^∞ and right coprime, and

$$M^*M + N^*N = I, \tag{7}$$

and that $G = \overline{M}^{-1}\overline{N}$ is a normalized left coprime factorization (NLCF) for G if $\overline{M}$, $\overline{N}$ are in H_+^∞ and left coprime and

$$\overline{M}\overline{M}^* + \overline{N}\overline{N}^* = I. \tag{8}$$

Let now K be an $m \times p$ rational function. We say that the pair (G, K) is *internally stable* if

$$\begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} \in H_+^\infty.$$

It is well known that the factorizations in H_+^∞ of the internally stabilizing controllers $K = UV^{-1} = \overline{V}^{-1}\overline{U}$ of G satisfy the Bezout equations

$$\overline{V}M - \overline{U}N = I \tag{9}$$

and

$$\overline{M}V - \overline{N}U = I \tag{10}$$

and that if (U_0, V_0) is one solution, then any other controller is obtained by the Youla parametrization

$$K = (U_0 + MQ)(V_0 + NQ)^{-1}, \qquad Q \in H_+^\infty. \tag{11}$$

The matrices $M, N, \overline{M}, \overline{N}, U, V, \overline{U}, \overline{V} \in H_+^\infty$ are called a doubly coprime factorization of a plant G if

$$\begin{bmatrix} \overline{M} & \overline{N} \\ \overline{U} & \overline{V} \end{bmatrix} \begin{bmatrix} V & -N \\ -U & M \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{12}$$

We quote now the following result from [15]:

THEOREM 2 (Vidyasagar and Kimura). *Let the pair (G, K) be internally stable, and suppose it has a doubly coprime factorization $M, N, \overline{M}, \overline{N}, U, V, \overline{U}, \overline{V}$. Let $\|\Delta_M, \Delta_N\|_\infty < \varepsilon$, and set $\tilde{G} = (\overline{N} + \Delta_{\overline{N}})(\overline{M} + \Delta_{\overline{M}})^{-1}$. Then the pair $(\tilde{G}, K)$ is internally stable if and only if*

$$\left\| \begin{matrix} U \\ V \end{matrix} \right\|_\infty < \varepsilon^{-1}.$$

The function G is, in general, neither stable nor antistable, and therefore it is not very interesting to consider it as a symbol of a Hankel operator; but it turns out that with G we can associate in a canonical manner an antistable function R_Λ^* to which we can apply the theory exposed above. This function R_Λ^* is obtained from a normalized coprime factorization (M, N) of G, using the following result from [6]:

THEOREM 3. *Let M, N and $\overline{M}, \overline{N}$ be right and left normalized coprime factorizations, respectively, of a rational plant G, and let $(U, V) \in H_+^\infty$ satisfy the Bezout equation $\overline{M}V - \overline{N}U = I$. Then the strictly proper antistable part R_0^* of $M^*U + N^*V$ is independent of the choice of U and V. Similarly, let $(\overline{U}, \overline{V})$ be a solution to the Bezout equation $\overline{V}M - \overline{U}N = I$. Then the strictly proper antistable part $\overline{R}_0^*$ of $\overline{U}\overline{M}^* + \overline{V}\overline{N}^*$ is independent of the choice of $\overline{U}$ and $\overline{V}$. Moreover, $R_0^* = \overline{R}_0^*$, and there exist unique pairs $U_0, V_0, \overline{U}_0, \overline{V}_0$ such that $K_0 = U_0 V_0^{-1} = \overline{V}_0^{-1}\overline{U}_0$ is stabilizing for G and*

$$R_0^* = M^*U_0 + N^*V_0 = \overline{U}_0\overline{M}_0^* + \overline{V}_0\overline{N}_0^*. \tag{13}$$

Still a direct consequence from [6] is the following:

PROPOSITION 1. *Let $R^* \in L^\infty$ be a rational function whose antistable strictly proper part is $R_0^* = M^*U_0 + N^*V_0 = \overline{U}_0\overline{M}_0^* + \overline{V}_0\overline{N}$. Then there exist unique pairs (U, V) and $(\overline{U}, \overline{V})$ in H_+^∞ satisfying the Bezout equations*

(9) *and* (10) *respectively, such that*

$$R^* = M^*U + N^*V = \overline{U}\overline{M}^* + \overline{V}\overline{V}^*.$$

Proof. From (11) we get

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \begin{bmatrix} M \\ N \end{bmatrix} Q.$$

Multiplying on the left by $[M^*, N^*]$, we get

$$M^*U + N^*V = M^*U_0 + N^*V_0 + (M^*M + N^*N)Q = R_0^* + Q.$$

Therefore, choosing $Q = R^* - R_0^*$ will yield the desired (U, V). ∎

3. MAIN RESULTS

We can now make a particular choice of R^* and (U, V) in view of Theorem 1. Given a function $G = NM^{-1} = \overline{M}^{-1}\overline{N}$, we can define R_Λ^* to be the superoptimal Nehari extension of $R_0^* = P_-(M^*U + N^*V)$, where (U, V) is any solution of the Bezout equation. Clearly, in view of Theorems 1 and 3, R_Λ^* is well defined and does not depend on the choice of the stabilizing controller (U, V). Define next $U_\Lambda, V_\Lambda \in H_+^\infty$ as the solution (existing and unique in view of Proposition 1) to

$$M^*U_\Lambda + N^*V_\Lambda = R_\Lambda^*. \tag{14}$$

With an argument similar to the above it can be seen that $\overline{R}_0^* := \mathbf{P}-(\overline{U}\overline{M}^* + \overline{V}\overline{N}^*)$ is independent o the choice of $(\overline{U}, \overline{V})$, it has a unique superoptimal Nehari extension $\overline{R}_\Lambda^*$, and there exists a unique pair $(\overline{U}_\Lambda, \overline{V}_\Lambda)$ which satisfies

$$\overline{U}_\Lambda \overline{M}^* + \overline{V}_\Lambda \overline{N}^* = \overline{R}_\Lambda^*. \tag{15}$$

Then we have:

LEMMA 1. *Let* (M, N) *and* $(\overline{M}, \overline{N})$ *be normalized right and left coprime factorizations of* G, *and let* $U_\Lambda, V_\Lambda, \overline{U}_\Lambda, \overline{V}_\Lambda$ *be defined from* (14) *and* (15). *Then* $R_\Lambda^* = \overline{R}_\Lambda^*$, $U_\Lambda V_\Lambda^{-1} = \overline{V}_\Lambda^{-1}\overline{U}_\Lambda$, *and* $U_\Lambda, V_\Lambda, \overline{U}_\Lambda, \overline{V}_\Lambda$ *satisfy the doubly coprime factorization* (12).

Proof. By construction $M, N, \overline{M}, \overline{N}$ are normalized coprime factorizations of the same G, so the only thing to see is that $R_\Lambda^* = \overline{R}_\Lambda^*$ and $U_\Lambda V_\Lambda^{-1} = \overline{V}_\Lambda^{-1}\overline{U}_\Lambda$. We have

$$\begin{bmatrix} M & \overline{N}^* \\ N & -\overline{M}^* \end{bmatrix}\begin{bmatrix} M^* & N^* \\ \overline{N} & -\overline{M} \end{bmatrix} = \begin{bmatrix} M^* & N^* \\ \overline{N} & -\overline{M} \end{bmatrix}\begin{bmatrix} M & \overline{N}^* \\ N & -\overline{M}^* \end{bmatrix} = I. \quad (16)$$

Then

$$\begin{aligned} \overline{V}_\Lambda U_\Lambda - \overline{U}_\Lambda V_\Lambda &= \left[\overline{V}_\Lambda, -\overline{U}_\Lambda\right]\begin{bmatrix} M & \overline{N}^* \\ N & -\overline{M}^* \end{bmatrix}\begin{bmatrix} M^* & N^* \\ \overline{N} & -\overline{M} \end{bmatrix}\begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} \\ &= \left[I, \overline{R}_\Lambda^*\right]\begin{bmatrix} R_\Lambda^* \\ -I \end{bmatrix}, \end{aligned}$$

i.e. $\overline{V}_\Lambda U_\Lambda - \overline{U}_\Lambda V_\Lambda = R_\Lambda^* - \overline{R}_\Lambda^*$. Since the left hand side of this equality is in H_+^∞, it means that R_Λ^* and $\overline{R}_\Lambda^*$ have the same antistable part; but since the superoptimal Nehari extension is unique, the result follows. ∎

The controller $K_\Lambda = U_\Lambda V_\Lambda^{-1} = \overline{V}_\Lambda^{-1}\overline{U}_\Lambda$ is called the *superoptimally robust stabilizing controller* of G. The name derived from the fact that (U_Λ, V_Λ) is the controller which not only provides the largest stability margin on the generic perturbations of the plant, but also gives greater margins (than the first one) for perturbations of (M, N) for which the component $\eta_1 \lambda_1 \xi_1^\#$ remains unchanged (or within the original stability margin). We refer the reader to [14] or [10] for details.

The next theorem extends to the multivariable case a result of [6].

THEOREM 4. *Let (ξ_k, η_k) be the Schmidt pairs of $H_{R_\Lambda^*}$, and σ_k the corresponding singular values. Then the Hankel operator $H_{\left[\begin{smallmatrix}\overline{M}^* \\ \overline{N}^*\end{smallmatrix}\right]}$ has singular values $\mu_k = \sigma_k / \sqrt{1 + \sigma_k^2}$ and Schmidt pairs $\{\xi_k, \hat{\eta}_k\}$, where*

$$\begin{aligned} \hat{\eta}_k &= \frac{1}{\sqrt{1+\sigma_k^2}}\left(\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k\right) \\ &= \frac{1}{\sqrt{1+\sigma_k^2}}\left(\begin{bmatrix} -N \\ M \end{bmatrix}\eta_k + \sigma_k\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\xi_k\right) \end{aligned}$$

and ϕ_k is as in (6).

Proof. In view of (12), (15), and (6), we can write

$$\begin{bmatrix} \overline{M} & \overline{N} \\ -N^* & M^* \end{bmatrix}\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k = \begin{bmatrix} R_\Lambda \\ I \end{bmatrix}\eta_k = \begin{bmatrix} \sigma_k \xi_k + \phi_k \\ \eta_k \end{bmatrix}$$

and

$$\begin{bmatrix} \overline{M} & \overline{N} \\ -N^* & M^* \end{bmatrix}\begin{bmatrix} \overline{M}^* - V_\Lambda \\ \overline{N}^* + U_\Lambda \end{bmatrix}\xi_k = \begin{bmatrix} 0 \\ R_\Lambda^* \end{bmatrix}\xi_k = \begin{bmatrix} 0 \\ \sigma_k \eta_k + \psi_k \end{bmatrix}.$$

Multiply on both sides by

$$\begin{bmatrix} \overline{M}^* & -N \\ \overline{N}^* & M \end{bmatrix}, \qquad \text{which is the inverse of} \begin{bmatrix} \overline{M} & \overline{N} \\ -N^* & M^* \end{bmatrix},$$

and subtract from the second equation the first multiplied by σ_k:

$$\begin{bmatrix} \overline{M}^* - V_\Lambda \\ \overline{N}^* + U_\Lambda \end{bmatrix}\xi_k - \sigma_k\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k = \begin{bmatrix} \overline{M}^* & -N \\ \overline{N}^* & M \end{bmatrix}\begin{bmatrix} -\sigma_k^2\xi_k - \sigma_k \phi_k \\ \psi_k \end{bmatrix},$$

or

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\xi_k(1 + \sigma_k^2) = \sigma_k\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \sigma_k\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k + \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix}\xi_k + \begin{bmatrix} -N \\ M \end{bmatrix}\psi_k,$$

where the first two terms in the second member are in H_-^2 and the other two are in H_+^2. Hence we have shown that

$$H_{\left[\begin{smallmatrix} \overline{M}^* \\ \overline{N}^* \end{smallmatrix}\right]}\xi_k = \frac{\sigma_k}{1 + \sigma_k^2}\left(\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k\right).$$

To conclude the first claim, observe that

$$[\overline{M}, \overline{N}]\left(\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k\right) = R\eta_k - \phi_k = \sigma_k \xi_k$$

and thus

$$H_{[\overline{M}, \overline{N}]}\frac{1}{\sqrt{1+\sigma_k^2}}\left(\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k\right) = \frac{\sigma_k}{\sqrt{1+\sigma_k^2}}\xi_k,$$

which is exactly what we wanted.

We would like to stress the fact that the equation obtained above and (23) are not symmetric, since the left hand side of (23) is in H_-^2, whereas the left hand side of the above equation is not in H_+^2. Therefore, to apply the standard argument to derive a Nevanlinna-Pick problem we will have to use (23).

To verify the second expression of $\hat{\eta}_k$, observe that (13) and (10) together with (16) yield

$$[\overline{U}_\Lambda, \overline{V}_\Lambda] = [R^*, I]\begin{bmatrix} \overline{M} & \overline{N} \\ -N^* & M^* \end{bmatrix}$$

and hence

$$\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} = \begin{bmatrix} \overline{M}^*R - N \\ \overline{N}^*R + M \end{bmatrix}.$$

From (6) we obtain $\phi_k = R_\Lambda^*\eta_k - \sigma_k \xi_k$; then

$$\begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\phi_k = \begin{bmatrix} \overline{M}^*R - N \\ \overline{N}^*R + M \end{bmatrix}\eta_k - \begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}R\eta_k + \sigma_k\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\xi_k$$

$$= \begin{bmatrix} -N \\ M \end{bmatrix}\eta_k + \sigma_k\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix}\xi_k. \qquad \blacksquare$$

The obvious dual of the previous result, with a similar proof, is the following:

THEOREM 5. *The Hankel operator with symbol* $H_{[M^*, N^*]}$ *has Schmidt pairs* $\{\hat{\xi}_k, \eta_k\}$, *where*

$$\hat{\xi}_k = \frac{1}{\sqrt{1+\sigma_k^2}}\left(\begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix}\xi_k - \begin{bmatrix} M \\ N \end{bmatrix}\psi_k\right)$$

$$= \frac{1}{\sqrt{1+\sigma_k^2}}\left(\begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}\xi_k + \sigma_k\begin{bmatrix} M \\ N \end{bmatrix}\eta_k\right),$$

with singular values $\mu_k = \sigma_k / \sqrt{1+\sigma_k^2}$, *where* ψ_k *is as in* (5).

LEMMA 2. *The functions* $U_\Lambda, V_\Lambda, \bar{U}_\Lambda, \bar{V}_\Lambda$ *satisfy the conditions*

$$U_\Lambda^* U_\Lambda + V_\Lambda^* V_\Lambda = Q(I + \Lambda^2)Q^*, \tag{17}$$

$$\left(\bar{U}_\Lambda \bar{U}_\Lambda^* + \bar{V}_\Lambda \bar{V}_\Lambda^*\right)\bar{Q}^* = \bar{Q}^*(I + \Lambda^2), \tag{18}$$

where Q *and* $\bar{Q}$ *are as in* (4).

Proof. As above, we can write

$$\begin{bmatrix} \bar{M} & \bar{N} \\ -N^* & M^* \end{bmatrix}\begin{bmatrix} \bar{U}_\Lambda^* \\ \bar{V}_\Lambda^* \end{bmatrix}\bar{Q}^* = \begin{bmatrix} R_\Lambda \\ I \end{bmatrix}\bar{Q}^* = \begin{bmatrix} Q\Lambda D \\ \bar{Q}^* \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{M} & \bar{N} \\ -N^* & M^* \end{bmatrix}\begin{bmatrix} \bar{M}^* - V_\Lambda \\ \bar{N}^* + U_\Lambda \end{bmatrix}Q = \begin{bmatrix} 0 \\ R_\Lambda^* \end{bmatrix}Q = \begin{bmatrix} 0 \\ Q^*\Lambda D^* \end{bmatrix}.$$

Multiply on both sides by

$$\begin{bmatrix} \bar{M}^* & -N \\ \bar{N}^* & M \end{bmatrix},$$

and subtract from the second equation the first multiplied on the right by $D^*\Lambda$:

$$\begin{bmatrix} \overline{M}^* - V_\Lambda \\ \overline{N}^* + U_\Lambda \end{bmatrix} Q - \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} \overline{Q}^* D^* \Lambda = \begin{bmatrix} \overline{M}^* & -N \\ \overline{N}^* & M \end{bmatrix} \begin{bmatrix} -Q\Lambda^2 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} Q(1 + \Lambda^2) = \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} \overline{Q}^* D^* \Lambda - \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix} Q$$

In conclusion,

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} = \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix} \overline{Q}^* D^* \Lambda (1 + \Lambda^2)^{-1} Q^* - \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix} Q(1 + \Lambda^2)^{-1} Q^*. \quad (19)$$

Multiplying (19) by $[V_\Lambda^*, -U_\Lambda^*]$, we obtain, in view of (10),

$$I = (U_\Lambda^* U_\Lambda + V_\Lambda^* V_\Lambda) Q (1 + \Lambda^2)^{-1} Q^*$$

and hence the first formula; multiplying (19) by $[\overline{U}_\Lambda, \overline{V}_\Lambda]$, we get

$$R_\Lambda^* = \overline{Q}^* \Lambda D^* Q^* = \left(\overline{U}_\Lambda \overline{U}_\Lambda^* + \overline{V}_\Lambda \overline{V}_\Lambda^*\right) \overline{Q}^* D^* \Lambda (1 + \Lambda^2)^{-1} Q^*,$$

or

$$\overline{Q}^*(1 + \Lambda^2) = \left(\overline{U}_\Lambda \overline{U}_\Lambda^* + \overline{V}_\Lambda \overline{V}_\Lambda^*\right) \overline{Q}^*,$$

as wanted. ∎

4. AN INVERSE PROBLEM

We have examined the optimally robust stabilizing controller of a given plant. We turn now to the inverse problem, that is, to characterize all plants stabilized by a given controller in a superoptimal manner.

A square rational matrix function $F \in H_+^\infty$ is said to be outer if it is a unit in H_+^∞. Define $S \in H_+^\infty$ as the (essentially) unique outer solution to

$$S^*S = U_\Lambda^* U_\Lambda + V_\Lambda^* V_\Lambda.$$

In view of (17), it is true that

$$Q^*S^*SQ = (I + \Lambda^2), \tag{20}$$

which implies that S is coercive and therefore a unit in H_+^∞, and therefore the vector

$$\begin{bmatrix} U_N \\ V_N \end{bmatrix} = \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} S^{-1}$$

clearly constitutes a normalized coprime factorization of the controller.

It is well known (see e.g. [12]) that if

$$W_1^* W_1 = W_2^* W_2 \tag{21}$$

with $W_1, W_2 \in H_+^\infty$ and W_1 outer, then there exists an inner function Q_1 such that

$$W_2 = Q_1 W_1.$$

Thus from (21) we get

$$SQ = Q_1 (I + \Lambda^2)^{1/2}$$

and hence

$$S = Q_1 (I + \Lambda^2)^{1/2} Q^*.$$

Since S is outer,

$$S^{-1} = Q(I + \Lambda^2)^{-1/2} Q_1^*$$

is in H_+^∞, and S^{-1} is therefore the outer factor of the outer-inner factorization of $Q(I + \Lambda^2)^{-1/2}$ and can be easily computer from Q and Λ.

Similarly, for $[\overline{U}_\Lambda, \overline{V}_\Lambda]$, we can definethe outer spectral factor $\overline{S} \in H_+^\infty$ as the solution to

$$\overline{S}\overline{S}^* = \overline{U}_\Lambda \overline{U}_\Lambda^* + \overline{V}_\Lambda \overline{V}_\Lambda^*$$

and obtain a left normalized coprime factorization of the controller:

$$\left[\overline{U}_N, \overline{V}_N\right] = \overline{S}^{-1}\left[\overline{U}_\Lambda, \overline{V}_\Lambda\right].$$

Again, in view of (18),

$$\overline{Q}\,\overline{S}\,\overline{S}^*\overline{Q}^* = I + \Lambda^2, \tag{22}$$

and using again the dual of (21), we obtain

$$\overline{Q}\,\overline{S} = (I + \Lambda^2)^{1/2}\overline{Q}_1$$

(where $\overline{Q}_1$ is now a rigid function, i.e. a $p \times m$ function in H_+^∞ such that $\overline{Q}_1\overline{Q}_1^* = I$). Again $\overline{S}$ is outer; that is,

$$I + \Lambda^{2-1/2}\overline{Q} = \overline{Q}_1\overline{S}^{-1},$$

and, as before, $\overline{Q}_1$ is the rigid factor of the rigid outer factorization of $\overline{Q}(I + \Lambda^2)^{-1/2}$.

So the equations relating plant and superoptimal controller now appear, in view of (19), as

$$\begin{aligned}
\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} &= \begin{bmatrix} \overline{U}_\Lambda^* \\ \overline{V}_\Lambda^* \end{bmatrix}\overline{Q}^*D^*\Lambda(I + \Lambda^2)^{-1}Q^* + \begin{bmatrix} V_\Lambda \\ -U_\Lambda \end{bmatrix}Q(I + \Lambda^2)^{-1}Q^* \\
&= \begin{bmatrix} \overline{U}_N^* \\ \overline{V}_N^* \end{bmatrix}\overline{S}^*\overline{Q}^*D^*\Lambda(I + \Lambda^2)^{-1}Q^* + \begin{bmatrix} V_N \\ -U_N \end{bmatrix}(S^*)^{-1}.
\end{aligned}$$

Multiplying by S^* and setting

$$T^* = \overline{S}^*\overline{Q}^*D^*\Lambda(I + \Lambda^2)^{-1}Q^*S^* = \overline{Q}^*D^*\Lambda Q_1^*,$$

we get finally

$$\begin{bmatrix} \overline{M}^* \\ \overline{N}^* \end{bmatrix} S^* = \begin{bmatrix} \overline{U}_N^* \\ \overline{V}_N^* \end{bmatrix} T^* + \begin{bmatrix} V_N \\ -U_N \end{bmatrix}. \tag{23}$$

Equation (23) has clearly a dual: the derivation is similar but not entirely trivial and is obtained in the following

LEMMA 3. *The following relation holds*:

$$\begin{bmatrix} M \\ N \end{bmatrix} \overline{Q}^* \overline{QS} = \begin{bmatrix} U_N \\ V_N \end{bmatrix} T + \begin{bmatrix} \overline{V}_N^* \\ -\overline{U}_N^* \end{bmatrix} \overline{Q}_1^* \overline{Q}_1. \tag{24}$$

Proof. In view of (12), (15) and (6), we can write

$$\begin{bmatrix} M^* & N^* \\ -\overline{N} & \overline{M} \end{bmatrix} \begin{bmatrix} M - \overline{V}_\Lambda^* \\ N + \overline{U}_\Lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ R_\Lambda \end{bmatrix}$$

and

$$\begin{bmatrix} M^* & N^* \\ -\overline{N} & \overline{M} \end{bmatrix} \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} = \begin{bmatrix} R_\Lambda^* \\ I \end{bmatrix}.$$

Multiply on both sides by

$$\begin{bmatrix} M & -\overline{N}^* \\ N & \overline{M}^* \end{bmatrix},$$

and subtract the second equation multiplied by R_Λ from the first. We get

$$\begin{bmatrix} M - \overline{V}_\Lambda^* \\ N + \overline{U}_\Lambda^* \end{bmatrix} - \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix} R_\Lambda = \begin{bmatrix} M & -\overline{N}^* \\ N & \overline{M}^* \end{bmatrix} \begin{bmatrix} -\overline{Q}^* \Lambda^2 \overline{Q} \\ 0 \end{bmatrix}.$$

Multiply on the right by $\overline{Q}^*\overline{Q}$ and rearrange terms:

$$\begin{bmatrix} M \\ N \end{bmatrix}\overline{Q}^*(I+\Lambda^2)\overline{Q} = \begin{bmatrix} \overline{V}_\Lambda^* \\ -\overline{U}_\Lambda^* \end{bmatrix}\overline{Q}^*\overline{Q} + \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix}R_\Lambda.$$

Multiplication on the right by $\overline{Q}^*(I+\Lambda^2)^{-1/2}\overline{Q}_1$ yields

$$\begin{aligned}
&\begin{bmatrix} M \\ N \end{bmatrix}\overline{Q}^*(I+\Lambda^2)^{1/2}\overline{Q}_1 \\
&= \begin{bmatrix} \overline{V}_\Lambda^* \\ -\overline{U}_\Lambda^* \end{bmatrix}\overline{Q}^*(1+\Lambda^2)^{-1/2}\overline{Q}_1 + \begin{bmatrix} U_\Lambda \\ V_\Lambda \end{bmatrix}QD\Lambda(1+\Lambda^2)^{-1/2}\overline{Q}_1 \\
&= \begin{bmatrix} \overline{V}_N^* \\ -\overline{U}_N^* \end{bmatrix}\overline{S}\overline{Q}^*(1+\Lambda^2)^{-1/2}\overline{Q}_1 + \begin{bmatrix} U_N \\ V_N \end{bmatrix}SQ(1+\Lambda^2)^{-1/2}D\Lambda\overline{Q}_1.
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
\begin{bmatrix} M \\ N \end{bmatrix}\overline{Q}^*\overline{QS} &= \begin{bmatrix} \overline{V}_N^* \\ -\overline{U}_N^* \end{bmatrix}\overline{Q}_1^*\overline{Q}_1 + \begin{bmatrix} U_N \\ V_N \end{bmatrix}Q_1D\Lambda\overline{Q}_1 \\
&= \begin{bmatrix} U_N \\ V_N \end{bmatrix}T + \begin{bmatrix} \overline{V}_N^* \\ -\overline{U}_N^* \end{bmatrix}\overline{Q}_1^*\overline{Q}_1,
\end{aligned}$$

as wanted. ∎

LEMMA 4. *$T^*T = \overline{S}^*\overline{S} - I$ and $TT^* = SS^* - I$.*

Proof. From (20) we have

$$\begin{aligned}
T^*T &= \overline{S}^*\overline{Q}^*D^*\Lambda(I+\Lambda^2)^{-1}Q^*S^*SQ(I+\Lambda^2)^{-1}\Lambda D\overline{Q}\,\overline{S} \\
&= \overline{S}^*\overline{Q}^*D^*\Lambda(I+\Lambda^2)^{-1}\Lambda D\overline{QS} \\
&= \overline{S}^*\overline{Q}^*\Lambda^2(I+\Lambda^2)^{-1}\overline{QS} \\
&= \overline{S}^*\overline{S} - I.
\end{aligned}$$

Similarly,

$$\begin{aligned} TT^* &= SQ(I+\Lambda^2)^{-1}\Lambda D\overline{Q}\,\overline{S}\,\overline{S}^*\overline{Q}^* D^*\Lambda(I+\Lambda^2)^{-1}Q^*S^* \\ &= SQ\Lambda DD^*\Lambda(I+\Lambda^2)^{-1}Q^*S^* \\ &= SQ\Lambda^2(I+\Lambda^2)^{-1}Q^*S^* \\ &= SS^* - I. \end{aligned}$$

Since, from (20), $L = SQ(I+\Lambda^2)^{-1/2}$ is inner, we obtain

$$TT^* = L\Lambda^2 L^*. \qquad \blacksquare \qquad (25)$$

In conclusion, given the normalized coprime factorizations of the plant, the superoptimal controller and all the functions T, S, $\overline{S}$ are uniquely determined.

We now consider, as promised, the inverse problem, i.e., given normalized coprime factorizations of the controller, what can be said about all the other functions occurring in (23)? Clearly the key point is to find the function T^*, since everything else is then uniquely determined. Now we want also (25) to be satisfied for a given Λ. Two questions arise: for which Λ (23) might have a solution, and how to compute it. The first question finds a simple answer in the following

LEMMA 5. *Equation* (23) *has a solution only if* $\Lambda \geqslant \Lambda_K$, *where* Λ_K *are the Young values of* $H_{U_N^* M + V_N^* N}$.

Proof. It is clear that there exists, in view of Theorem 3 applied to (U, V) instead of (M, N), a unique superoptimal Nehari extension R_K of $\mathbf{P} - (U_N^* M + V_N^* N)$ with Young values Λ_K. Therefore, from the very definition of superoptimal Nehari extension, $s_i(R(i\omega)) \geqslant s_i(R_K(i\omega))$, and therefore we reach the conclusion. $\blacksquare$

The next result is about the reduction of the two block interpolation problem (23) to a one block problem.

LEMMA 6. *Let* $T^* \in L^\infty$ *satisfy*

$$\overline{N}^* S^* = \overline{V}_N^* T^* - U_N \qquad (26)$$

for some $\bar{N}, S \in H_+^\infty$ *with* S *outer. Then* T^* *also satisfies*

$$\bar{M}^* S^* = \bar{U}_N^* T^* + V_N \tag{27}$$

for some $\bar{M} \in H_+^\infty$, *and therefore satisfies* (23). *Moreover,* $\bar{M}, \bar{N}, S$ *can be chosen so that* $(\bar{M}, \bar{N})$ *are left normalized coprime.*

Proof. Let P be the minimal degree inner function (denote this degree by n_K) such that $U_N P^*, V_N P^* \in H_-^\infty$, and denote by (s_j, v_j), $j = 1, \dots, n_K$, the zeros of P^* [i.e. the pairs (s_j, v_j) for which $P^*(S_j)v_j = 0$]. Then (23) is equivalent to

$$\left[\left(\bar{U}_N^* T^* + V_N\right) P^*\right](s_j) v_j = 0,$$

$$\left[\left(\bar{V}_N^* T^* - U_N\right) P^*\right](s_j) v_j = 0, \qquad j = 1, \dots, n_K.$$

Deriving $T^* P^*(s_j) v_j$ from the second equation, we get

$$T^* P^*(s_j) v_j = \left[\left(\bar{V}_N^*\right)^{-1} U_N P^*\right](s_j) v_j;$$

substitution in the first yields

$$\left[\bar{U}_N^* \left(\bar{V}_N^*\right)^{-1} U_N P^*\right](s_j) v_j = -[V_N P^*](s_j) v_j$$

and thus

$$\left[(V_N^*)^{-1} U_N^* U_N P^*\right](s_j) v_j = -[V_N P^*](s_j) v_j,$$

or

$$[(U_N^* U_N + V_N^* V_N) P^*](s_j) v_j = 0.$$

But this is always verified, and thus so is (27). ∎

In conclusion, we need to solve (26) under the condition

$$TT^* = Q_1 \Lambda^2 Q_1^*. \tag{28}$$

Let P^* be, as in the proof of the preceding lemma, the Douglas-Shapiro-Shields factor of $\begin{bmatrix} U_N \\ V_N \end{bmatrix}$. Then (28) is equivalent to

$$L^* = T^*P^* = L_1^*\Lambda L_2^* \tag{29}$$

with the condition that L_2 is inner and L_1 is rigid. Then, computing (26) in the zeros of P^*, we obtain, as above,

$$0 = [\bar{N}^*S^*P^*](s_i)v_i = \left[\left(\bar{V}_N^*T^* - U_N\right)P^*\right](s_i)v_i$$

and thus

$$\left[\bar{V}_N^*T^*P^*\right](s_i)v_i = [U_N P^*](s_i)v_i.$$

Suppose now that $\bar{V}_N(s_i)$ is invertible for $i = 1, \dots, n_K$. Then we can write

$$L^*(s_i)v_i = [T^*P^*](s_i)v_i = \left(\bar{V}_N^*\right)^{-1}(s_i)[U_N P^*](s_i)v_i,$$

and this, with the conditions (29), is "almost" a Nevanlinna-Pick problem, in the sense that the only change with respect to the usual formulation is the factor Λ. So we try now to rewrite our equations in terms of a standard Nevanlinna-Pick problem (in fact, it turns out that we have to consider a Schur-Takagi problem). We define the set

$$\mathcal{N}_{u,v,s} := \{u_i \in \mathbb{C}^p, v_i \in \mathbb{C}^m, s_i \in \mathbb{C}; i = 1, \dots, n_K\}$$

Consider the matrix P given by

$$p_{ij} = \left[\frac{u_j^* u_i - v_j^* v_i}{1 - s_i \bar{s}_j}\right]_{i,j=1,\dots,n_K},$$

and suppose it has k negative eigenvalues. It is well known (see e.g. [2]) that then there exists a rational unitary function of degree n_K, with at most k *stable* poles, which satisfies the interpolating conditions. We therefore say that the set $\mathcal{N}_{u,v,s}$ is a set of Schur-Takagi data. In particular, if $P \geqslant 0$ we say that $\mathcal{N}_{u,v,s}$ is a set of Nevanlinna-Pick data.

PROPOSITION 2. *Let $L \in H_+^\infty$ be an $m \times p$ rational function such that*

$$L^* = L_1^* \Lambda L_2^* \tag{30}$$

where L_1 is an $m \times p$ rigid function ($L_1 L_1^* = I_p$) and L_2 is a $p \times p$ inner function. Suppose L^* satisfies the interpolating conditions

$$L^*(s_i)u_i = v_i. \tag{31}$$

Let $\hat{L}^*$ be a rigid, not necessarily stable solution to the Schur-Takagi problem

$$\hat{L}^*(s_i)\Lambda^{-1}u_i = v_i.$$

Suppose $\hat{L}_1^*$ is a left factor of $\hat{L}^*$, and let w_i be solutions to

$$\hat{L}^*(s_i)w_i = v_i.$$

Then, if the Pick matrices

$$p_{ij} = \left[\frac{u_j^* u_i - w_j^* w_i}{1 - s_i \bar{s}_j}\right]_{i,j=1,\dots,n_K}$$

and

$$p_{ij} = \left[\frac{w_j^* \Lambda^2 w_i - v_j^* v_i}{1 - s_i \bar{s}_j}\right]_{i,j=1,\dots,n_K}$$

are nonnegative definite, then any pair of solutions L_1 and L_2 with Nevanlinna-Pick data $\mathcal{N}_{u,w,s}$ and $\mathcal{N}_{\Lambda w,v,s}$ satisfies (31). *Conversely, if the spectrum of Λ is simple, any L^* satisfying* (31) *and* (30) *has a unique representation only if L_1^* and L_2^* have no diagonal factors on the right and on the left, respectively.*

Proof. That L satisfies the interpolation conditions follows from the construction. To prove the second statement, suppose

$$L_1^* \Lambda L_2^* = \tilde{L}_1^* \Lambda \tilde{L}_2^*$$

where $L_1, L_2, \tilde{L}_1^*, \tilde{L}_2^*$ are inner. Then, multiplying both sides by the transposed conjugate, we obtain

$$L_1^* \Lambda^2 L_1 = \tilde{L}_1^* \Lambda^2 \tilde{L}_1,$$

or

$$\Lambda^2 L_1 \tilde{L}_1^* = L_1 \tilde{L}_1^* \Lambda^2,$$

and thus $L_1 \tilde{L}_1^*$ is diagonal. ∎

In conclusion, we have a description of all the plants whose superoptimal robust stabilizing controller is K.

THEOREM 6. *Let (U_N, V_N) and $(\overline{U}_N, \overline{V}_N)$ be right and left normalized coprime factorizations of a controller K, and suppose $\overline{V}_N^*$ is nonsingular in the poles of U_N. Then for each $\Lambda \geqslant \Lambda_K$ the set of plants G which are optimally robustly stabilized by K with Young values Λ is given by*

$$G = \left(\overline{U}_N^* T^* + \overline{V}_N\right)^{-1}\left(\overline{V}_N^* T^* - U_N\right),$$

where T satisfies the conditions

$$\overline{V}_N^* T^* - U_N \in H_-^2 \tag{32}$$

and

$$TT^* = Q_1 \Lambda^2 Q_1^*. \tag{33}$$

Let P be the minimal inner function such that

$$\begin{bmatrix} U_N \\ V_N \end{bmatrix} P^* \in H_-^2.$$

For $i = 1, \dots, n_K$, let (u_i, s_i) be such that $P^*(s_k)u_i = 0$ (we assume the zeros of P^* are simple) and $v_i = [(\overline{V}_N^*)^{-1} U_N P^*](z_i)v_i$. Then all the T satisfying (32) *and* (33) *are given by*

$$T^* = L^* P^*$$

where L^* is as in Proposition 2 *with Nevanlinna-Pick data $\mathcal{N}_{u,v,z}$.*

There is a canonical way to parametrize all the solutions to the above Nevanlinna-Pick problems, and we refer to [3] for details.

Remark that the above is not a parametrization, since the same T can be obtained in different ways if the factorization of $L^* = T^*P^* = Q_1^*\Lambda Q_2^*$ is not unique. Nevertheless, as we showed in Proposition 2, this case is nongeneric.

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